# CALCULUS OF VARIATIONS, PARTIAL DIFFERENTIAL EQUATIONS, AND GEOMETRY

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## Contents



## Summary

The purpose of this chapter is to illustrate on elementary and classical examples the strong interaction between geometry, calculus of variations and the analysis of partial differential equations. We will show in particular that partial differential equations can be used to solve problems of geometrical nature, and that some partial differential equations, introduced first as models in physics present some truly geometrical features, which in turn may lead to unexpected solutions to problems in geometry.

## 1. Introduction

### 1.1 Generalities

The strong connection between calculus of variations and geometry goes back presumably to the very beginning of these fields. A very natural question: “Is it possible to find a two-dimensional domain with least area, for a given prescribed parameter?”(as in the isoperimetric problem) offers a good example of such an interplay. With the discovery of analytic geometry (that is, the use of coordinates in order to describe geometric objects) and the invention of differential calculus, it was soon recognized that many of these kind of geometric problems could be formulated very efficiently (even though not necessarily in the most elegant terms) as differential equations. It opened the way to a systematic treatment of a large number of questions involving first one-dimensional objects (essentially lines). In the modern period, some important progresses made in the understanding of partial differential equations (PDE’s) allow us also to successfully consider higher dimensional problems. A classical example, which is rather simple to formulate, is the minimal surface problem.

In another direction, the growing geometrization of physics, whose natural language is, for a large part, differential calculus, tightened the links described above even more strongly. Here are some examples, of rather different natures:

**-Soap films:** Take a thin metal wire, twist it so to obtain a ring-shaped object, and dip it into water containing soap. A film of soap will form, whose boundary is exactly the ring. The surface formed that way corresponds (at least at first approximation) to a minimal surface.

**-Curvature driven interfaces:** These problems, in contrast with the previous one which was a stationary problem, involve evolution in time. A typical example, which also corresponds to some familiar experiment, is the melting of ice in water. If we dip an ice cube into water, it will melt, and hence its shape will evolve in time. The boundary of the cube evolves according to an equation involving its curvature (actually its mean curvature). A similar equation (involving this time the Gaussian curvature) appears in models for eroding rocks at the bottom of the ocean.

**-The Einstein equations of General Relativity:** These equations describe the evolution of our universe considered as a geometric object. The equations governing this evolution involve the fact that the space is “curved” (in some abstract sense), a notion which is modeled by the precise geometrical concept of “curvature”. In good coordinates (the choice of which is extremely subtle) these equations have striking analogies with the equation for a vibrating string, or that of sound (or electromagnetic) waves, which have been studied extensively since the late eighteenth century. Even very basic questions such as the well-posedness of these equations (i.e. does the solution exist and is it unique?), or the formation of singularities present tremendous difficulties. It is only in the last decades that spectacular progresses have been made, but the theory is still far from being complete.

Geometric evolution equations of the above kind (and specially the second example) play an important role not only in physics, but arise also in models in economy or biology. They turn out to be also an impressive tool in solving fundamental questions of purely geometrical (or topological) nature. The best example is presumably the recent solution to the Poincaré conjecture by G. Perelman, using the so-called “Ricci flow”, another curvature evolution equation. It is also worth mentioning that related ideas have been used in a completely different direction for image processing: geometric evolution equations are used there to deform or select those curves, which carry the most significant or relevant information in a given image.

In the previous description, we have emphasized the role of partial differential equations as a **tool** in some geometric problems. At a later stage of the discussion, we will also show that for some important PDE’s, properties of solutions have to be described by geometrical means, and that, sometimes, they even shed new light on the geometric concepts they are related to.

### 1.2 Parameterization of Geometrical Problems

The first important step in order to provide an analytical treatment of a geometrical problem consists in its parameterization. One has to describe the geometric object with suitable coordinates (in some topics, the term “gauge” is more appropriate), such that the unknowns are now “functions” (of real variables), and such that the problem for these functions can be handled using the tools provided by the existing theory for PDE's (or tools that could be developed for that purpose).

In general there is no unique way to construct a parameterization. Moreover, a parameterization may introduce some new difficulties, which are not genuinely related to the original question: we will see in particular on our examples, that parameterization might introduce some new **invariance** by a large group, which is **not present** in the original problem! From the analytical point of view, this might represent a serious source of trouble, related to so-called non-compactness.

Another important point we wish to stress is that, as far as possible, a parameterization should preserve some remarkable features of the problem. An illustration of this assertion is provided by the **maximum principle**. It can be formulated for instance for minimal surfaces (see below) as follows: if two minimal surfaces are such that (in some suitable sense) their boundaries lie one over the other, then the same property holds in the interior, i.e. one surface should be above the other. A good parameterization should preserve a similar property for the functions aimed at describing these surfaces, and we will see that this is indeed the case for graphs, and offers moreover an efficient method to solve the problem.

**Remark 1.** *We would like to mention, however that, in order to solve some geometrical problems, approaches avoiding* **parameterization by functions** *have also been developed. Most of them proceed from the so called "Geometric Measure Theory". In some sense, this theory provides a framework similar to functional analysis, but for objects which are meant to be "lines", "surfaces", geometrical objects in general, in some rather crude sense. As in classical functional analysis, these spaces are designed to have nice properties (local compactness, for instance), so that many arguments from functional analysis can be transferred.*

We illustrate next the concepts introduced so far with a classical example, which played an important historical role in the development of the field.

## 2. An Example: Minimal Surfaces

We consider in this section the classical plateau problem for minimal surfaces, which corresponds to our first example, the soap film. In mathematical terms, it can be stated as the following problem. Let $\gamma$ be a smooth closed curve in $\mathbb{R}^3$, with no self-intersection:

(Q) *Is there a surface M minimizing the area among all surfaces with boundary* $\gamma$ *?*

This simple question has led to a great variety of approaches. We start with the one which is presumably conceptually the simplest.

### 2.1 Graphs

Assume that in some appropriate coordinates the curve $\gamma$ can be represented as a graph, i.e. that there exists a two-dimensional simply connected domain $\Omega$ in $\mathbb{R}^2$ such that $\Sigma \equiv \partial\Omega$ is a smooth closed curve, and such that there exists a parameterization $\tilde{g} : \Sigma \to \gamma$ of $\gamma$ of the form $\tilde{g}(\sigma) = (\sigma, g(\sigma))$, where $g$ is a smooth real-valued function on $\Sigma$. In this context, it is natural to look for a solution to (Q) having itself the form of a graph over $\Omega$, i.e. of the form $\tilde{f}(x) = (x, f(x))$, for $x = (x_1, x_2) \in \Omega$, where *f* is a real-valued function on $\Omega$. In this case, the correspondence between the unknown function *f*, and the surface represented by *f* (i.e. its graph), is one to one, an optimal situation, which is unfortunately not always met, as we will see in the next subsection. The area of the surface parameterized by $\tilde{f}$ is given by the formula

$$A(f) = \int_\Omega \sqrt{1 + |\nabla f(x_1, x_2)|^2}\, dx_1 dx_2.$$

A minimizer *f* for *A* is a critical point of *A*, that is, the first variation of *A* at *f* vanishes. This means that for every function $\psi$ with compact support in $\Omega$, we have

$$\frac{dA}{ds}(f + s\psi)\Big|_{s=0} = -\int_{\mathcal{O}} \operatorname{div}\left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) \psi = 0.$$

Since the choice of test function $\psi$ is arbitrary, we obtain the nonlinear partial differential equation, also called the **Euler-Lagrange** equation for the functional *A*

$$\begin{cases} \operatorname{div}\left( \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} \right) = 0 & \text{in } \Omega. \\ f = g & \text{on } \gamma. \end{cases} \tag{1}$$

Notice that Eq. (1) expresses the fact that the mean curvature of the graph is exactly zero (the left hand side being precisely the expression of half the mean curvature of a graph). Therefore (1) is presumably one of the first and simplest “curvature equations”.

In order to find solutions to the previous problem, it is tempting to attack the minimization problem directly: it offers some nice specificities, in particular the functional *A* is strictly convex, so that if a minimizer exists, it should be unique, and also be the unique solution to the Euler-Lagrange equation associated to the problem. However, in order to find such a minimizer one has to enlarge the quest for a suitable functional space, having the appropriate topological properties, and on which the functional is well defined. It turns out that the space adapted to the functional *A* is the space $BV(\Omega)$ of generalized functions of bounded variations, that is, functions whose derivatives, in the sense of distributions, are bounded measures. In this space, we are hence able to find the ‘desired” solutions. However, *BV* functions may be rather discontinuous, for instance they may have a jump across a full curve. Therefore, in order to complete the program it remains to establish higher regularity properties for the

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**Biographical Sketch**

**Fabrice Bethuel** was born June 7, 1963.
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